Circular Error in Price Index Numbers
Based on Scanner Data. Preliminary Interpretations.

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Abstract

Differences between chain and base methods (or strategies) are analyzed. This is a preliminary paper that concentrates on numerical results and new ideas. Omitted proofs will appear in later papers. As in title, we use the term circular or chain *error*. We do not use milder terms as *drift* or *deviation* in the identity or circular test, because it is definitely an error of the chain index. Chain error is a norm of all proper index numbers formulas (except Lowe, Jevons and their derivatives which are not proper) and it vanishes only in special circumstances. Our main claim is that chain error is almost zero only if *all* its links are described by the *same* almost ideal demand theory. This is demonstrated for a comprehensible set of both contingently biased and excellent index numbers.

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1 Introduction

The value of a price index number depends on three things: data in question, the strategy used (base, chain or rather a mixture of them) and on the index number formula. These should fit together and give maximally reliable results, see Vartia (2018). Scanner data makes possible to experiment with different combinations of these, e.g. frequent chaining. The key problem of the chain type of indices is the *chain error* that tends to grow when chaining is applied frequently – typically on a monthly basis.

False assumptions are often made of index number formulas or of the data on which they are based. For example, the main mistakes of Fisher (1922) were assuming that Laspeyres and Paasche indices were almost equal and ´very good´ indices and that the variances of price and quantity log-changes are essentially the same, see Vartia & Suoperä, 2018. A non-technical summary of the good choices for complete data is Vartia (2018).

We compare the base and the chain strategy using several known index number formulas. The base strategy is our benchmark because it is always free of the chain error. We use especially the Multi Period Identity Test (MPIT), where the price-links make a circle and return to the initial point. Their product should equal unity.

The magnitude of chain error depends on the nature of data and index number formula used. For the chained type strategies, we define the circular time path such that $0 \rightarrow 1 \rightarrow 2 \rightarrow \ldots \rightarrow T \rightarrow 0$. In this case the price index series should equal unity. If not, the chain error exists. We see that the time path consists of separate links in order. We call them *direct price-links*. As such price-links are non-problematic, but their multiplication (or chaining) causes the probable chain error. We argue like Fisher (1922) that some chain error appears normally for all proper[2] index number formulas and chained type strategies - sometimes it is harmless and sometimes serious. The algebraic treatment of chain error seems too difficult. Our main point is that the chain or circular error is approximately zero for excellent formulas, only if all direct price-links come from the *same* almost ideal consumer demand theory. If the chain error is large, the same demand theory cannot describe the price-links.

When prices and quantities adjust nicely hand in hand according to demand theory, the chain error essentially vanishes and all strategies and all excellent index number formulas are practically equal.

In chapter two we present data and index number formulas used in this paper. In chapter three we present various strategies for measuring price changes and for constructing index series. In chapter four relevant tests (i.e. CT, MPIT, TRT, PIT) are presented. Chapter five present empirical results of the study and chapter six concludes.

2 Test Data, Notations and Index Number Formulas

2.1 Description of Test Data

We use a scanner data from one big Finnish retail trade chain that contains information on three chained shops; ‘hypermarkets’, ‘supermarkets’ and ‘small outlets’. The test data set contains monthly data from three years 2014, 2015 and 2016 collected from five regions and five different item categories as classified by the enterprise; ‘Fresh meat’, ‘Fresh fish’, ‘Milk products’, ‘Cheese’ and ‘Eggs’. These enterprise-based categories include commodities belonging to 16 different coicop5 commodity groups. We classify our data as cartesian product of regions, size of outlets, coicop5 commodity groups and EAN commodity identifiers. For each category the data set contains price and quantity information and the data may called as complete micro data. This data includes about 22 000 commodities that are comparable in quality in all time periods.

[2] Only for fixed weight formulas Lowe and Jevons and their derivatives the chain error is always zero. These are not proper index number formulas, because either commodity reversal test CRT or unit of measurement test UMT are not satisfied – except for Jevons with unity weights.

2.1 Notation

Our notation for the index number calculations follows Vartia & Suoperä, 2018.

2.3 Index Number Formulas

Index number formula is defined accurately in Vartia 2010 and Nieminen & Montonen, 2018. We use a simple notation here:

$$(p^0,\ q^0, p^1,\ q^1) \rightarrow\ P_n(p^0,\ q^0, p^1,\ q^1) = P_n{}^{1/0}$$

Consider some price index number formula *P* comparing period 0 to period 1 i.e. $P^{1/0}$. We say that a price-link from a period 0 to the period 1 is defined. The price-link is a *direct binary comparisons* based on an index number formula from 0 to 1, in this direction. The point in the link is in the *periods* for which the calculation is made. These may vary, while the index number formula *P* stays normally the same. It is important to decide, what periods are compared directly using index number formula and for what periods the changes are calculated indirectly using these previously calculated direct figures. The former directly calculated index numbers just define our strategy for constructing the index series. To stress our point, the links say for which periods the price index formula is applied. All other calculations are derived ones. The choice of links determines the strategy of construction of index series.

We analyze two sets of index number formulas. The first set is based on formulas using old or new weights. Laspeyres (*L*), Log-Laspeyres (*l*) and Harmonic Laspeyres (*Lh*) uses base period weights (i.e. old weights) and Palgrave (*Pl*), Log-Paasche (*p*) and Paasche (*Pa*) instead uses observation period weights (i.e. new weights). We call these index number formulas as *basic formulas*. The second set of index numbers include six formulas: Stuvel (*St*), Montgomery-Vartia (*MV*), Törnqvist (*T*), Fisher (*F*), Sato-Vartia (*SV*) and Walsh-Vartia (*W*). We call these index number formulas as *excellent formulas*[3]. The fundamental analysis of these index number formulas, see Vartia & Suoperä (2018). This paper uses this study as handbook and utilizes its results widely.

3 Various Strategies for Constructing Index Series

Estimates for the price changes depends on data in question, on index number formula used in calculation and on the strategy. The index number theory defines two main strategies - the base and chain strategies. In the base strategy the direct price-links are based on comparisons 0 → *t*, *t* = 1, 2, …, while in the chain strategy the direct price-links are *t*-1 → *t*, *t* = 1,2, … . These two are the simplest strategies to construct index series and actual strategies are normally some mixtures or combinations of them. Now notice, how price-links are basic elements on which the strategy of constructing price index series depend on. Note also, that any strategy can be chosen totally independently of the choice of index number formula *P*. The index number formula *P* and the strategy (choice of links) are two fundamental decisions in the index number construction. This almost self-evident distinction has been poorly treated and it is mostly lacking in index number literature, see e.g. ILO (2004) and compare Vartia (1976, 2010). Next, we define strategies used in in this paper.

[3] Includes superlative and other index number formulas, whose bias = 0 for small changes (Vartia and Suoperä 2018).

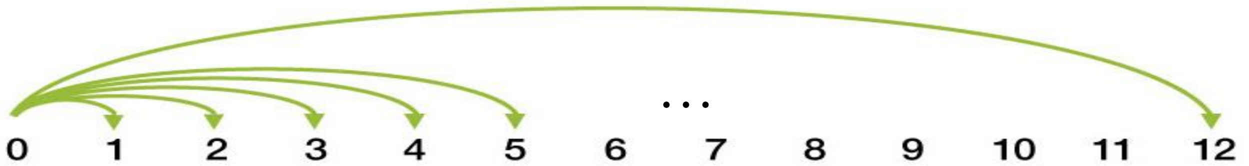
3.1 Base Strategy

Suppose that we have $(p^t,\ q^t)$ two *N*-vectors of prices and quantities that are strictly positive all time periods (i.e. $t = 0, \dots, T$). In the base strategy for any price index number *P* the direct price-links are based on comparisons $0 \rightarrow t$, $t = 0, 1, 2, \dots,$

(1) $$P^{0/0} = 1; \quad P^{1/0} = P(p^0,\ q^0, p^1,\ q^1); \dots \quad P^{T/0} = P(p^0,\ q^0, p^T,\ q^T)$$

We are interested in measuring price change between base and observation periods. As we see, the price change in any compilations (0, *t*) depends on $(p^0,\ q^0, p^t,\ q^t)$ four *N*-vectors of prices and quantities. These price changes are based on direct price-links and are binary comparisons between $0 \rightarrow t$, $t = 0, 1, 2, \dots, T$. We call them as *measured price changes*. The price change between (*T*-1, *T*) may be calculated by simple division of $P^{T/0}$ by $P^{T-1/0}$. This price index depends on six *N*-vectors of prices and quantities from time periods (0, *T*-1, *T*). These price changes are called as *derived price changes*.

The base strategy can be visualized by following graph ($t = 0, 1, 2, \dots, 12$)



It is easily noticed that each separate arrow forms direct price-link from the base period 0 to *t*. This price-link simply tells us in which time periods the binary compilation is evaluated by some index number formula *P*.

Fisher (1922 p. 312) preferred the base strategy: first it is simpler to conceive and to calculate, and second it means something clear and definite to everybody and third it has no cumulative error (or chain error). These arguments are extremely important, but unfortunately Fisher did not know, that the base strategy cannot be recommend for all index number formulas. Vartia & Suoperä (2018) shows that index number formulas based on merely old (i.e. $P_L^{t/0}, P_l^{t/0}, P_{Lh}^{t/0}$) or new value shares (i.e. $P_{Pl}^{t/0}, P_p^{t/0}, P_{Pa}^{t/0}$) are always *contingently biased* and the size of the bias depends on data in question. Vartia & Suoperä (2018) concluded that these formulas should never be used for the complete micro data. On the contrary *excellent index number formulas* and the base strategy gives superb results and should be used always when possible (Vartia & Suoperä, 2018).

The base strategy has been criticized due to one major disadvantage: as new products appear and old products disappear, it becomes increasingly difficult to match items between base and observation periods (i.e. missing matched pairs). The question now arises: should *the comparison month* and *the base month* be adjacent months (chain strategy) or should *the base month be fixed* (base strategy, ILO, 2004; p. 407)?

If we select a certain month as base period, we probably maximize the problem of new and disappearing products. List of commodities become too narrow at certain month and serious missing matched pairs are realized. It seems reasonable to prefer chained indices over fixed base indices (ILO, 2004; p. 407).

In this study, we show that selecting previous year as the base period (i.e. normalize previous year as average month) the situation is much better for the base strategy. This strategy is based on links like; *year* → *year.m*, *m* = 1, 2, …, 12 and year → year + 1 (e.g. 2015 → 2016.*m*, *m* = 1, 2, …, 12 and 2015 → 2016). Our empirical analysis show that this base strategy beats clearly the chain strategy. It includes more matched pairs in all links compared to chain strategy.

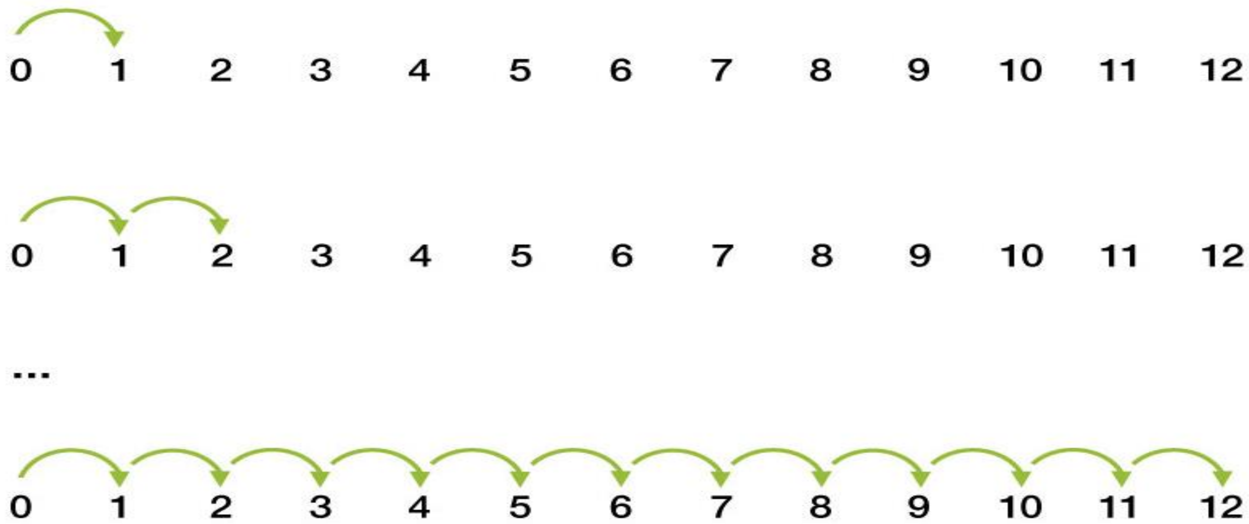
3.2 Chain Strategy

The second strategy is called as the chain strategy. We have $(p^t,\ q^t)$ two *N*-vectors of prices and quantities that are strictly positive all time periods (i.e. $t = 0,\ldots,T$). In the chain strategy for any price index number *P* the chain indices can be constructed as

(2)
$$
\begin{aligned}
&P^{0/0} = 1; \quad P^{1/0} = P(p^0,\ q^0, p^1,\ q^1) \\
&P^{2/0} = P(p^0,\ q^0, p^1,\ q^1)P(p^1,\ q^1, p^2,\ q^2) \\
&P^{T/0} = P(p^0,\ q^0, p^1,\ q^1)P(p^1,\ q^1, p^2,\ q^2) \ldots\ P(p^{T-1},\ q^{T-1}, p^T,\ q^T)
\end{aligned}
$$

Thus fixed base and chained price levels coincide for the first period but in the subsequent periods *t*, the fixed base indices compare the prices in period *t* directly to the prices in period 0, whereas the chained indices simply update the price level in the previous period by multiplying the period over period chain index by $P(p^{t-1},\ q^{t-1}, p^t,\ q^t)$. Now for the chain strategy, we notice that the links $P(p^{t-1},\ q^{t-1}, p^t,\ q^t)$ are direct price-links (i.e. is binary comparison between adjacent months) and we call them as *measured price changes*. But price changes 0 → *t*, $t \geq 2$ in the chained strategy are always *derived price changes* depending on all prices and quantities from all included time periods.

The chain strategy can be visualized by following graph (t = 0, 1, 2, .., 12)



In the chain strategy the direct price-links are applied to adjacent time periods. The measured price change is calculated by chosen price index number formula - the same for all direct price-links – and index series results simply by multiplication of them. Not the price-links as such but their multiplication is the source of the chain error. In this study we show that the basic index number formulas in the chain strategy *should not be applied*, because they include almost all the time severe chain error. Some chain error appears also for excellent formulas - not as severe as in basic indices - but chain error exists all the time. In the empirical part we see that the chain strategy leads to *contingent chain error*. This error happens ‘here and there’ and is independent of index number formula used. It is especially serious for the basic formulas. Our interpretation is that all price-links are simply not compatible with each other’s and cannot came from the same consumer demand theory. Some special commodities make on exception, where chain error happens to be negligible.

4 Various Tests for Different Strategies

In this paper we use only *time reversal* (TRT), *circular* (that is transitivity) (CT), *multi period identity* (MPIT) and *path independence test* (PIT). CT, MPIT and PIT are all needed to *reveal that the chain error occurs when an index does not return to unity when prices in the current period return to their levels in the base period.*

We classify index number formulas into two categories. **The first category** includes formulas that uses either old or new value shares. These basic formulas are *contingently biased* and do not satisfy the time reversal test (see Vartia & Suoperä, 2018).

We test these basic formulas comparing chain and base strategies for each index number formula separately. We simply calculate chained indices for any price index number formula *P* for time path $0 \rightarrow 1 \rightarrow 2 \rightarrow \ldots \rightarrow t \rightarrow 0$ for all $t = 1, 2, \ldots$ and the base indices for links $0 \rightarrow 1$, $0 \rightarrow 2$, $0 \rightarrow 3$, $\ldots 0 \rightarrow t$. Because the direct price-link or binary compilations (i.e. $0 \rightarrow t$ or $t \rightarrow 0$) have no circular or chain error, then if chained indices for any time path deviates from corresponding direct price-link, the chain strategy includes chain or cumulative error surely.

For any index number formula *P* (same formula for all direct price-links in order) *Circular Test* (CT) may be expressed as:

$$
\begin{aligned}
& P^{1/0}P^{2/1} = P^{2/0}\ (?) \\
& \qquad\vdots \\
(4) \qquad & \\
& P^{1/0}P^{2/1} \ldots P^{t/(t-1)} = P^{t/0}\ (?)
\end{aligned}
$$

These equations state that the chain and direct base indices should equal each other. No index number formula (except fixed base formulas Jevons and Lowe, which are not *proper* index number formulas) satisfy this test exactly for all data. If the estimates of these tests are approximately equal (i.e. left side chain index equals the right hand side direct index calculated by the same formula), the chain error is harmless, otherwise serious. In empirical analysis we calculate both sides and compare them, but we do not report the results in this paper. Instead we report more informative chain test, MPI test, for all formulas including the basic formulas (see definition of MPIT in eq. (6)).

The second category includes all *excellent index number formulas*. We analyze these index number formulas and show how their numerical values depends on data and on strategies used for constructing index series. These formulas satisfy *the time reversal test* i.e.

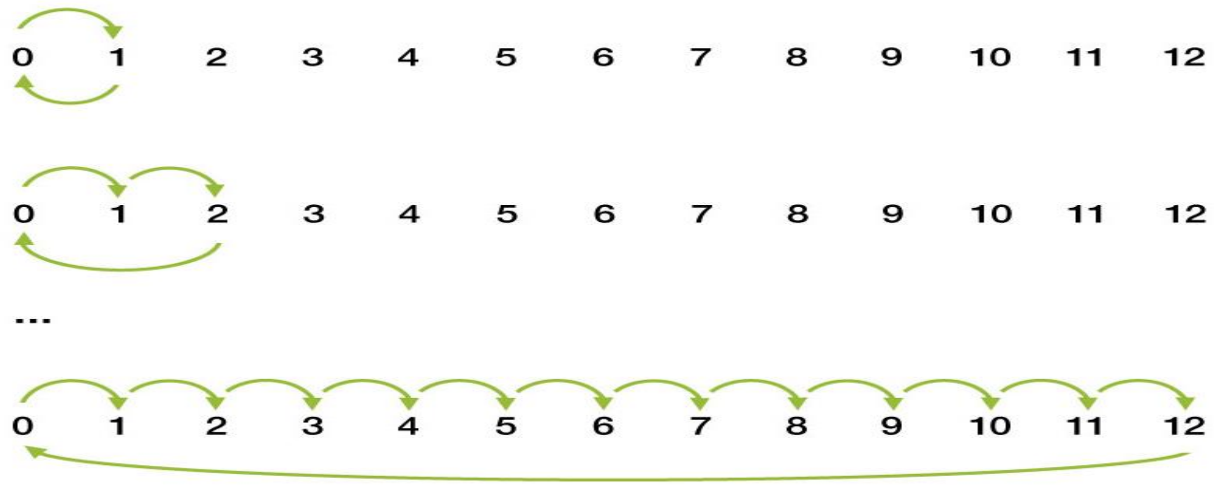
$$(5) \qquad P(p^0, q^0, p^1, q^1) = 1/P(p^1, q^1, p^0, q^0)$$

that states, that the price index number for period 1 relative to period 0 is the reciprocal of the price index number for period 0 relative to period 1. In other words, an index number formula *P* coincide its time antithesis (=TA, see, Fisher, 1922, p.118; Vartia, 1976, p. 73-75). The time antithesis (TA) property is analyzed nicely for the most known index number formulas in Vartia & Suoperä (Table 1, 2017). The same paper tells also how to use the time antithesis in rectifying formulas to get ‘*excellent’* results.

The third test that will be used is the multi period identity test, MPIT (Walsh, 1901, 1921). The test may be expressed for the chain strategy for time periods $t = 0, 1, 2, \ldots, 12$ and for the same formulas (including formulas that do not satisfy TRT) as

$$
\begin{aligned}
&P^{1/0}P^{0/1} = 1\ (?) \\
&P^{1/0}P^{2/1}P^{0/2} = 1\ (?) \\
(6)\qquad &\quad\vdots \\
&P^{1/0}P^{2/1} \ldots P^{12/11}P^{0/12} = 1\ (?)
\end{aligned}
$$

and is visualized for the chain strategy by the following graph (Month = 0 is the average month from previous year):

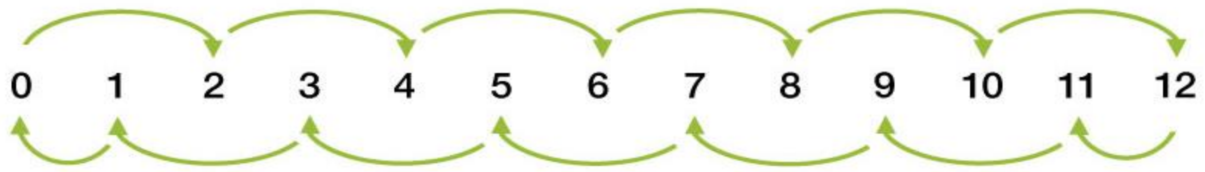


Similarly, as in case of the CT, any excellent formulas do not satisfy this test exactly. If the estimates of the MPIT's are approximately near to unity, the chain error is harmless, otherwise serious. In empirical analysis we show examples from both cases.

The last test is based on the path independence, where we define reasonable time path of calculation to reveal the circular error of the chained indices. We call this test as *the path independence test for chained strategies*. One possible test for example for 13 period window may be defined by the time path $0 \to 2 \to 4 \to 6 \to 8 \to 10 \to 12 \to 11 \to 9 \to 7 \to 5 \to 3 \to 1 \to 0$. We calculate all binary indices by any index number formula and chain or link the measured price changes together according to time path, i.e.

$$(8)\qquad P^{2/0}P^{4/2}P^{6/4}P^{8/6}P^{10/8}P^{12/10}P^{11/12}P^{9/11}P^{7/9}P^{5/7}P^{3/5}P^{1/3}P^{0/1} = 1\ \ (?)$$

and graphically



This time path starts from the base period 0 and goes via even months to December (T = 12) and then back via uneven months to base period. Alike in the ILO principle, start and end prices are equal and the chained strategy should equal unity. If it does not, chained system includes *circular error* and the chained strategy may not be used for this kind of datasets.

These chain error tests forms circular time paths:

1. The proper chain strategy starts from the base period (i.e. initial values), forms direct price-links for adjacent months in order and then forms direct price-link back to base period (i.e. for time path $0 \rightarrow 1 \rightarrow 2 \rightarrow \ldots \rightarrow t \rightarrow 0$ for all t = 1, 2, ...). This will be done for all time periods t = 1, 2, … . Evaluating these price-links by the same price index number formula and chaining these in order and direction should equal unity.
2. The path independence test starts from the base period 0 forms direct price-links between even months (i.e. $0 \rightarrow 2$, $2 \rightarrow 4$, …, $10 \rightarrow 12$) and then forms direct price-links back via uneven months to base period (i.e. $12 \rightarrow 11$, $11 \rightarrow 9$, …, $1 \rightarrow 0$). Evaluating these price-links by the same price index number formula and chaining these in order and direction should equal unity.

The tests above, are used to evaluate chain error for the chained type strategies, but these tests have more profound information about consumer demand theory. To stress our point, the *links* (see graphical presentations) say for which the price index formula is applied. Choice of links determines the *strategy* of construction of the index series. Now back to consumer demand theory: We may say that price-link is based on *ideal demand theory* or shortly is *ideal* if it is based on equilibrium points $\hat{q}^k = h\left(p^k, \hat{V}^k\right)$, ($k$ = 0, 1) of some demand theory, which is either (Vartia,1983)

(1) homothetic (homothetic price-link) or

(2) its points come from the same indifference surface ($\hat{q}^0 \sim \hat{q}^1 \sim \hat{q}^*$) (indifferent price-link)

In case of an ideal price-link, we are observing price indices from this kind of ideal demand theory. This terminology helps us to express theorems about circular properties of index numbers (see circular time paths from graphs for chained type strategies). For simplicity we take only three time periods and construct the circular chain or time path $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$, which includes three price-links.

Theorem 1: If any two links in the circular time path $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$ are ideal, then also the third link is ideal.

Proof: Suppose the links $0 \rightarrow 1$ and $1 \rightarrow 2$ are ideal. Then the same ideal demand theory governs all the three periods and also the link $2 \rightarrow 0$ is ideal.

Theorem 2: In the case of ideal links $P^{1/0}$, $P^{2/1}$ and $P^{0/2}$ the economic indices in the ideal demand system satisfy the circular test $P^{1/0}P^{2/1} = P^{2/0}$:

1. Homothetic links: $P(p^1, p^0)P(p^2, p^1) = \frac{c(p^1)}{c(p^0)}\frac{c(p^2)}{c(p^1)} = \frac{c(p^2)}{c(p^0)} = P(p^2, p^0)$
2. Indifferent links: $P(p^1, p^0; q^*)P(p^2, p^1; q^*) = P(p^2, p^0; q^*)$.

These can also be presented as $P_0^1 P_1^2\, P_2^0 = 1$:

3. Homothetic links: $P(p^1, p^0)P(p^2, p^1)P(p^0, p^2) = 1$.
4. Indifferent links: $P(p^1, p^0; q^*)P(p^2, p^1; q^*)\, P(p^0, p^2; q^*) = 1$.

These are well-known equalities for homothetic links, but less known for equilibrium points from the same indifference surface, cf. Vartia (1983).

This is not realized accurately for actual price indices, but only approximately. The four most important causes of approximation instead of exactness are:

1. Replacement of the economic price index by its empirical representation or approximation.
2. Approximation of the equilibrium quantities by their observed empirical quantities.
3. Approximation of an exact superlative index by some excellent formula providing a quadratic approximation.
4. Approximation of the unknown demand system, for which the superlative index and its demand system provide only “a flexible functional form”.
   When these are taken into account, the exact theoretical relations above appear only in approximate form $P_0^1 P_1^2\ P_2^0 = 1 + \epsilon$ for empirical price indices. The conclusion of all this is: Excellent price indices contain normally more or less circular or chain error. The chain error remains necessary small only if the three links are described by the *same* almost ideal demand theory.

If we chain together two price-links (i.e. 0 → 1, 1 → 2), which equals or approximate closely with the link 2 → 0, we may say, that these three links are compatible with each other. We may say, that they all are *ideal links* i.e. they all are in harmony with the same consumer demand theory. We apply circular time paths for multi time periods and test empirically whether these price-links are based on the same ideal demand theory or not? If they are not compatible with each other, then the *circular or chain error* exists and all price-links are not based on the same ideal demand theory all the time. Next, we show some empirical tests by graphs.

5 Empirical Results

The data used in this study is more closely described in chapter 2.1 Description of Test Data. Data includes about 22 000 commodities that are comparable in quality. The data contains price (unit prices) and quantity (total quantities sold in certain month) information needed in index calculation for all commodities and time periods in years 2014-2016.

First question for construction of index series is the selection of the strategy. We classify the strategies into two category; *base* or *chained type strategies*. Normally the base strategy is not recommended. Diewert and Fox (2017, p 8) give reason for that:

> “The question now arises: should the comparison month and the base month be adjacent months (thus leading to chained indices) or should the base month be fixed (leading to fixed base indices)? It seems reasonable to prefer chained indices over fixed base indices for two reasons:
> •The set of seasonal commodities which overlaps during two consecutive months is likely to be much larger than the set obtained by comparing the prices of any given month with a fixed base month (like January of a base year). Hence the comparisons made using chained indices will be more comprehensive and accurate than those made using a fixed base.
> • In many economies, on average 2 or 3 percent of price quotes disappear each month due to the introduction of new commodities and the disappearance of older ones. This rapid sample attrition means that fixed base indices rapidly become unrepresentative and hence it seems preferable to use chained indices which can more closely follow marketplace developments.” ILO (2004; 407).

Are these two arguments sufficient and reasonable causes to the exclusion of the base strategy? We show that they are not. The exclusion of the base strategy is caused by selection of bad strategy i.e. selecting a given month, like January, as a fixed base month. This surely did not promote the selection of the base strategy.

Instead, if we select previous year as the base period and normalize it as average month, the situation is completely different. This strategy effectively includes always more matched pairs compared to the proper chain strategy. In Figures 4.1 and 4.2 we represent empirical results for two base years 2014 and 2015. In both cases the base strategy clearly includes more matched pairs compared to the chain strategy all the time.

Our opposite empirical finding, compared to ILO opinion about the base strategy, is based on different *construction strategy* – ILO suggests *the base month* (i.e. January, February ...) and we the base year or its *normalized average month strategy*. If separate months have different consumption constructions then comparing two months excludes monthly specific commodities from the binary compilations.

Instead, the normalized average month strategy is based on the consumption construction from the previous year and so avoids effectively the exclusion of the monthly specific commodities. According to our data, the base strategy constructed using previous year as base period is difficult to beat by other strategies. This tells us that separate months have their specific seasonal features (i.e. *non-ideal links*), whom the chain strategy (excludes all monthly specific commodities) cannot take into account.

Figure 4.1: Number of matched pairs for base and proper chain strategies in year 2015. The base period for the base and the chain strategies is normalized average month from the year 2014.

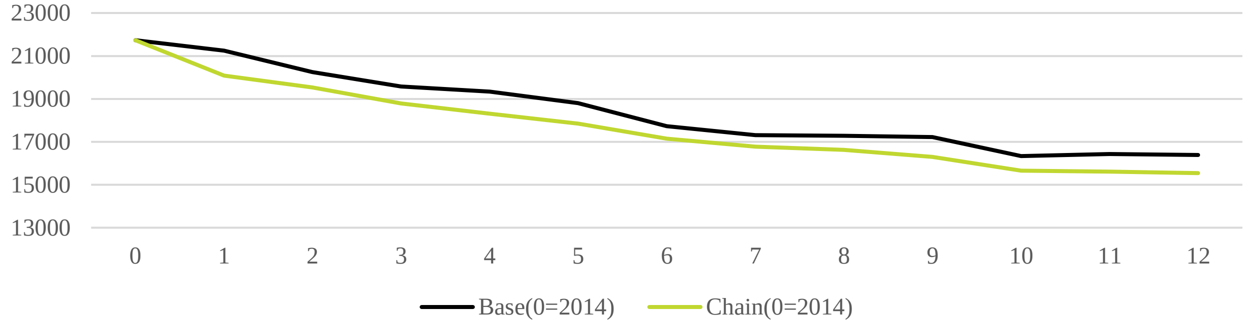
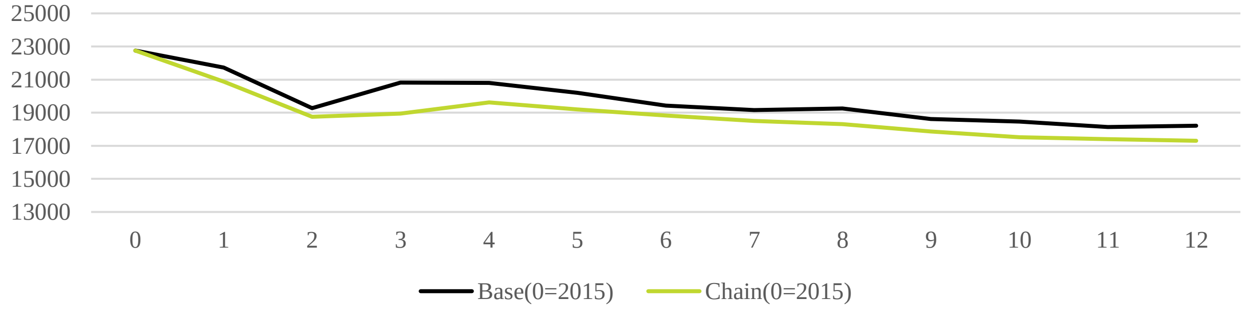


Figure 4.2: Number of matched pairs for base and proper chain strategies in year 2016. The base period for the base and the chain strategies is average month from the year 2015.



In this study we compare two strategies (base and chain) and twelve index number formulas [4]. We test *the circular test* (CT) and *the multi period identity test* (MPIT) for all index number formulas. All tests are aggregated from observation level into two categories:

1. Coicop5 groups (16) and
2. Coicop5 groups in five regions (16*5=80).

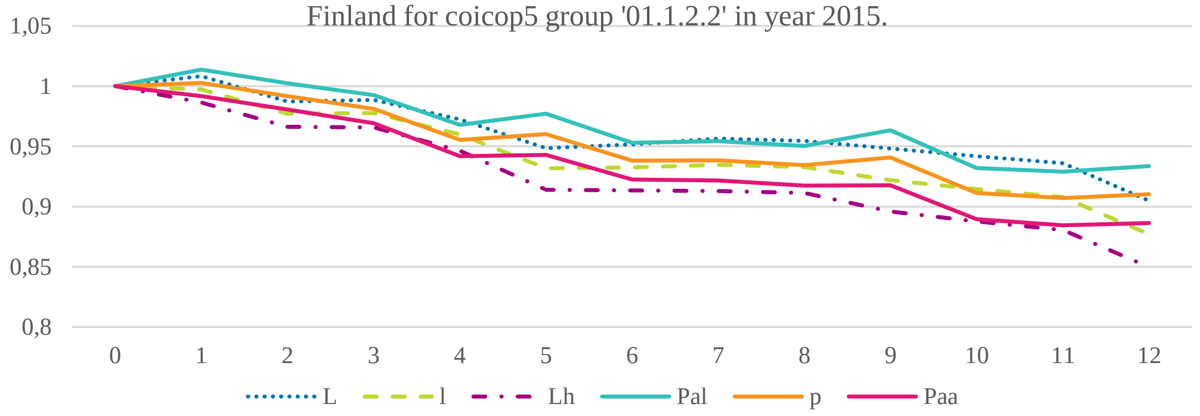
The tests reveal that the chain error occurs when an index does not return to unity when prices in the current period return to their levels in the base period (see the ILO, 2004, p. 445). All tests have been constructed for all time periods. When these tests deviate from unity we see the period when the chain error appears and, of course, if this chain error will stay in index series permanently. As have been noted, our benchmark strategy is the base strategy, because it does not contain this error.

4.1 Circular paths and MPIT results for the basic indices

We apply the chain strategy for the basic index number formulas i.e. for Laspeyres (*L*), Log-Laspeyres (*l*), Harmonic Laspeyres (*Lh*), Palgrave (*Pal*), Log-Paasche (*p*) and Paasche (*Pa*) formulas. We classify them in to two classes – harmless chain error (chain error less than 0.5 %) and severe chain error (chain error greater than 0.5 %). This division is done separately for 16 coicop5 commodity groups and for 16 coicop5 commodity groups in five regions. Results are then tabulated and are listed in Appendix 1.

Any of the 96 tests do not satisfy the circular path exactly, but we have tabulated them in to two categories – harmless (chain error less than 0.5%) and its complement severe chain error category. Tests show that 68 and 77 groups have severe chain error in years 2015 and 2016 respectively. The chain error is *data contingent* and particular severe for the basic index number formulas.

Figure 4.3: The chain error for basic index number formulas for chain strategy in Finland for coicop5 group '01.1.2.2' in year 2015.



Figures shows that all test series deviates dramatically from unity – less than unity means downward chain error and above unity upward chain error. Also, different coicop5 groups have different 'Fisher forks' in 2015 and 2016. The size of chain error for the same commodity groups differs in 2015 and 2016. Vartia and Suoperä (2018) shows that these basic index number formulas are *contingently biased*. The old and new forks appear in the same way as in Vartia & Suoperä (2018) – e.g. Laspeyres and Paasche are not equal as Fisher (1922)

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[4] Laspeyres = *L*, Log-Laspeyres = *l*, Harmonic Laspeyres = *Lh*, Palgrave = *Pl*, Log-Paasche =*p*, Paasche = *Pa*, Stuvel = *S*, Montgomery-Vartia = *MV*, Törnqvist = *T*, Walsh-Vartia = *W*, Sato-Vartia = *SV* and Fisher = *F*

erroneously assumed. How these three-forks based on old and new values shares are related to each other (or how biased the indices are) depends on data or is a contingent phenomenon. These index number formulas applied for chain strategy include also very severe *contingent chain error*. They are simply so unreliable formulas that they should never be used for any strategies, if excellent indices are available.

Figure 4.4: The chain error for basic index number formulas for chain strategy in Finland for coicop5 group '01.1.2.2' in year 2016.

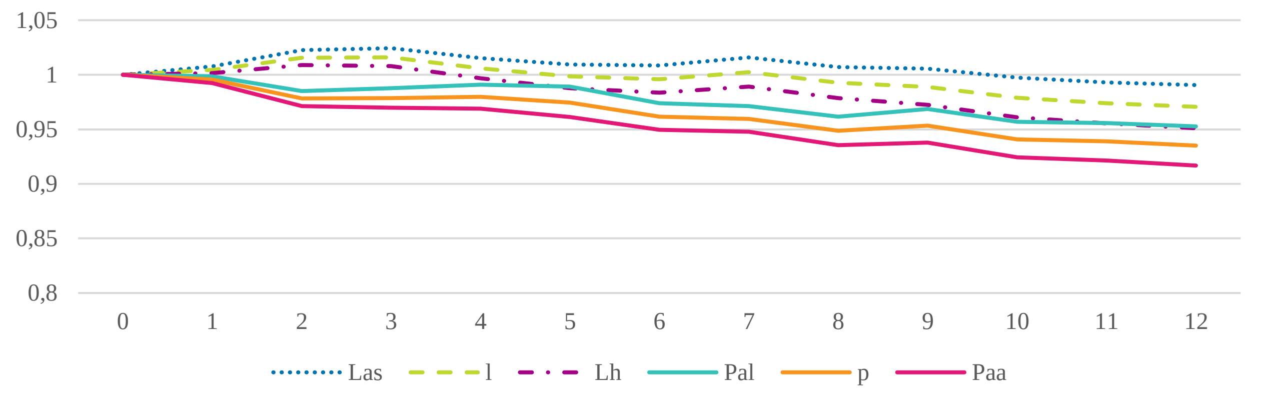


Figure 4.5: The chain error for basic index number formulas for chain strategy in Finland for coicop5 group '01.1.3.1' in year 2015.

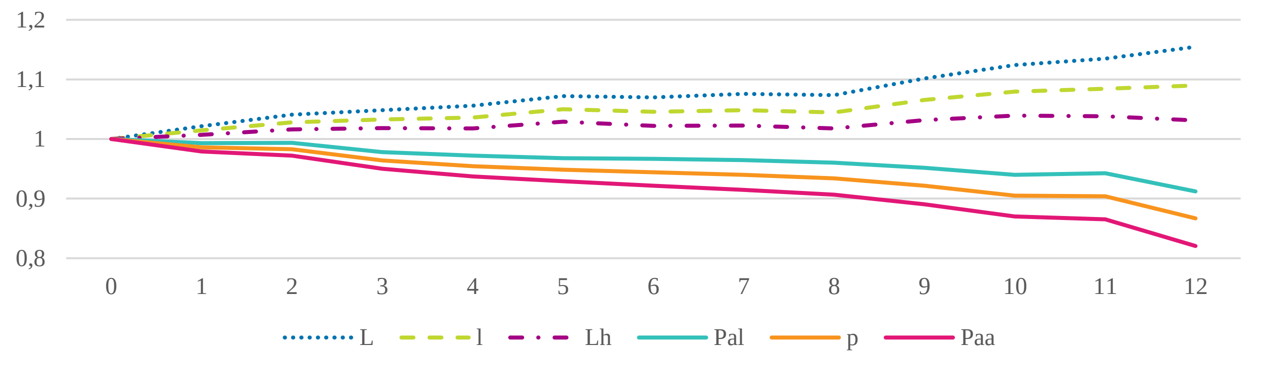
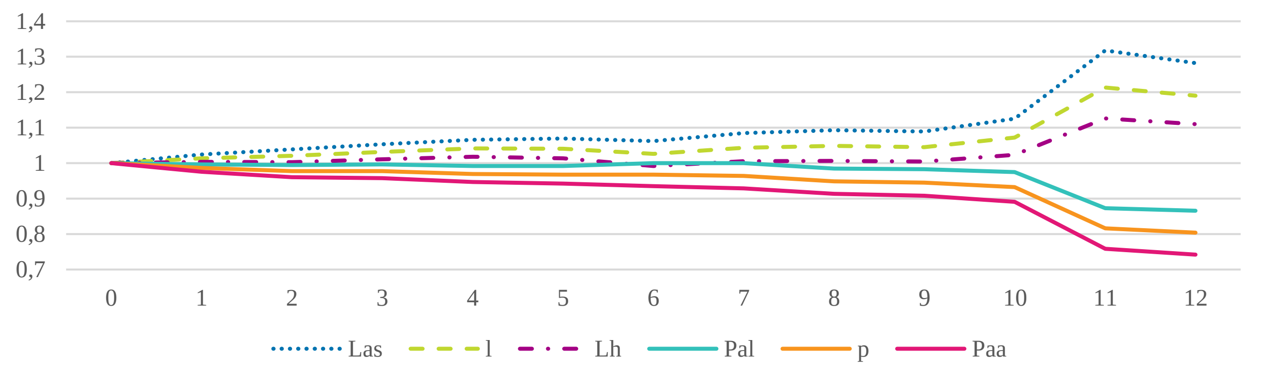


Figure 4.6: The chain error for basic index number formulas for chain strategy in Finland for coicop5 group '01.1.3.1' in year 2016.



4.2 MPIT results for excellent formulas

Here we have applied the MPIT for the proper chain strategy and for the excellent index number formulas. The results are reported for Stuvel (*S*), Montgomery-Vartia (*MV*), Törnqvist (*T*), Walsh-Vartia (*W*), Sato-Vartia (*SV*) and Fisher (*F*) formulas.

4.2.1. MPIT for the proper chain strategy in years 2015 and 2016

The MPIT's for the excellent formulas and chain strategy should equal unity for all time periods. In these years 63 and 59 commodity groups from 96 (about 65 %) have severe chain error (i.e. error greater than 0.5 %), harmless ones are presented in Appendix 1. We show some groups graphically. Note how the excellent indices move together near each other. They seem to have systematically negative chain errors, which becomes larger as December (12) is approached. It is clear, that the same demand theory cannot describe the price-links.

Figure 4.7: The chain error for excellent index number formulas for chain strategy in Finland for coicop5 group '01.1.2.1' in year 2015.

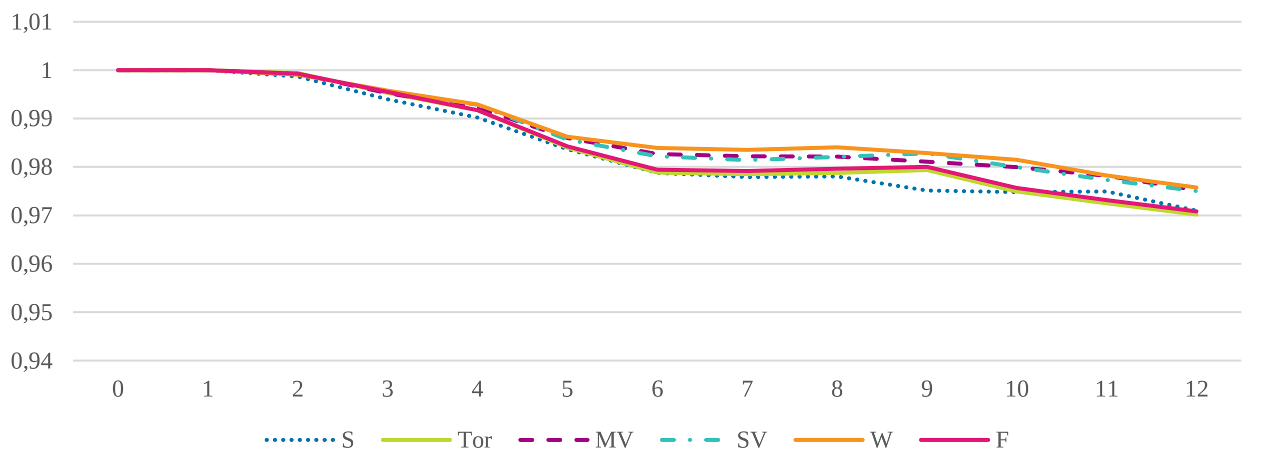
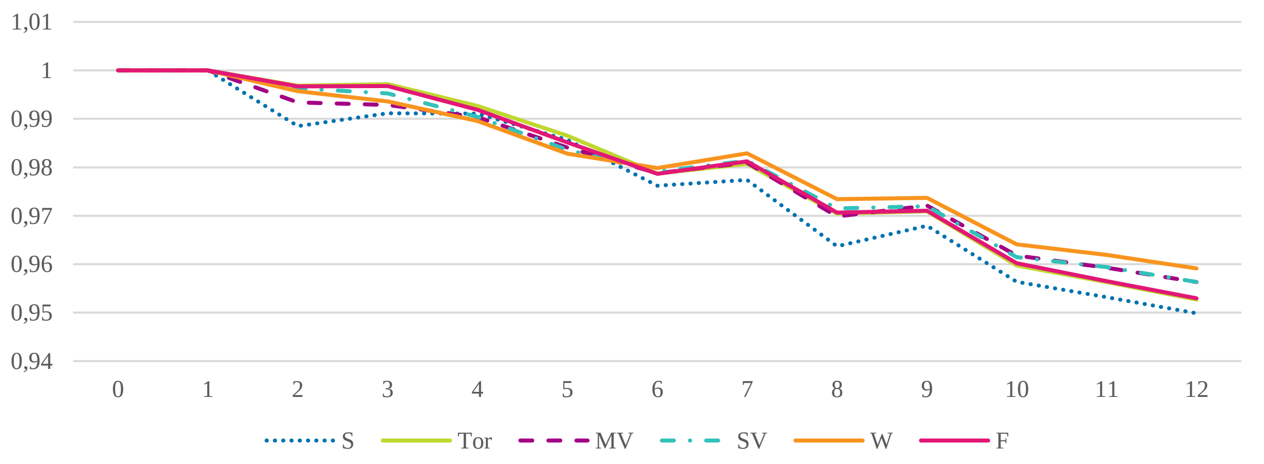


Figure 4.7b: The chain error for excellent index number formulas for chain strategy in Finland for coicop5 group '01.1.2.1' in year 2016.



Figures show for some groups that downward *contingent chain error* appears for chain strategy. It reveals that separate direct price-links do not come from the same consumer demand theory and the permanent negative chain error appears in the index series.

Figure 4.8: The chain error for excellent index number formulas for chain strategy in Finland for coicop5 group '01.1.2.2' in year 2015.

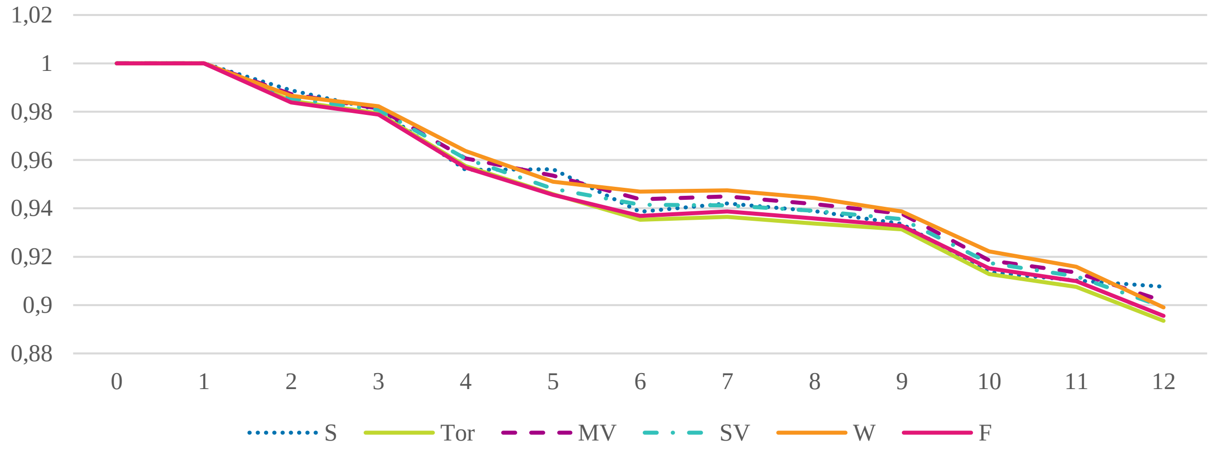
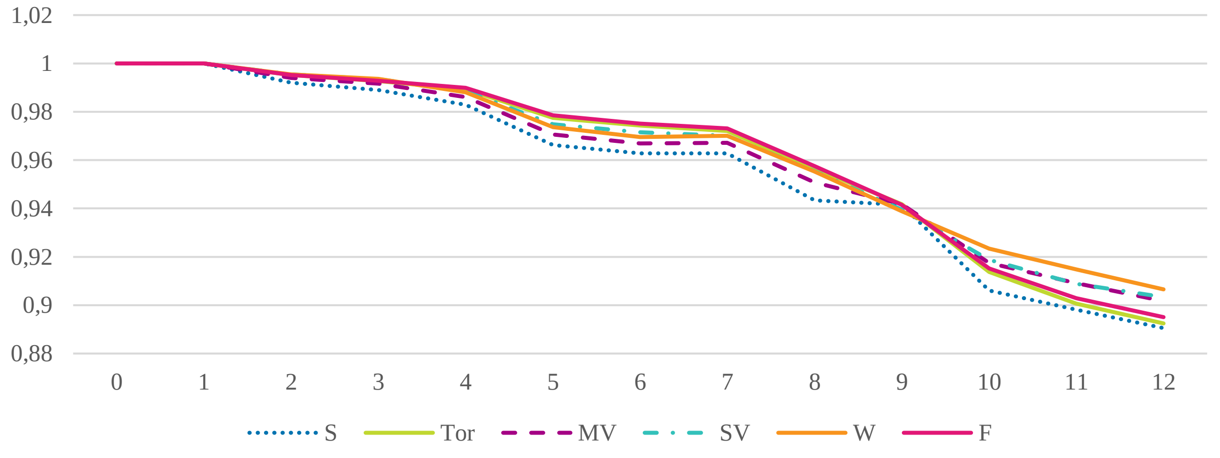


Figure 4.9: The chain error for excellent index number formulas for chain strategy in Finland for coicop5 group '01.1.2.2' in year 2016.



4.2.2 Path Independence Tests for the Basic Formulas in years 2015 and 2016

We apply the chain strategy for the basic index number formulas i.e. Laspeyres (*L*), Log-Laspeyres (*l*), Harmonic Laspeyres (*Lh*), Palgrave (*Pal*), Log-Paasche (*p*) and Paasche (*Pa*) formulas for the time path 0→2→4→6→8→10→12→11→9→7→5→3→1→ 0. As in the ILO principle, start and end prices are equal and the chained strategy should equal unity when all price-links have been evaluated and index series updated just the same order as in the time path shows. In 2015 and 2016 69 and 65 commodity groups from 96 have severe chain error (i.e. index series deviate more than 0.5 % form unity), harmless ones are presented in Appendix 1.

Now December is located at point 6. It is typical that the chain error of the basic formulas scatter much as the point 13 is approached. The conclusion is, that the basic formulas suffer from strong contingent chain error for the commodity groups shown.

Figure 4.10: PIT for basic index number formulas for chain strategy in Finland for coicop5 group '01.1.2.2' in year 2015.

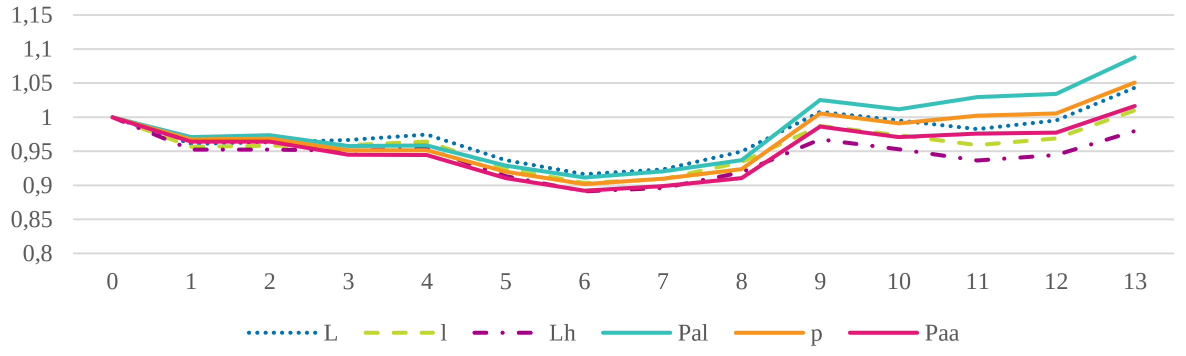


Figure 4.11: PIT for basic index number formulas for chain strategy in Finland for coicop5 group '01.1.2.2' in year 2016.

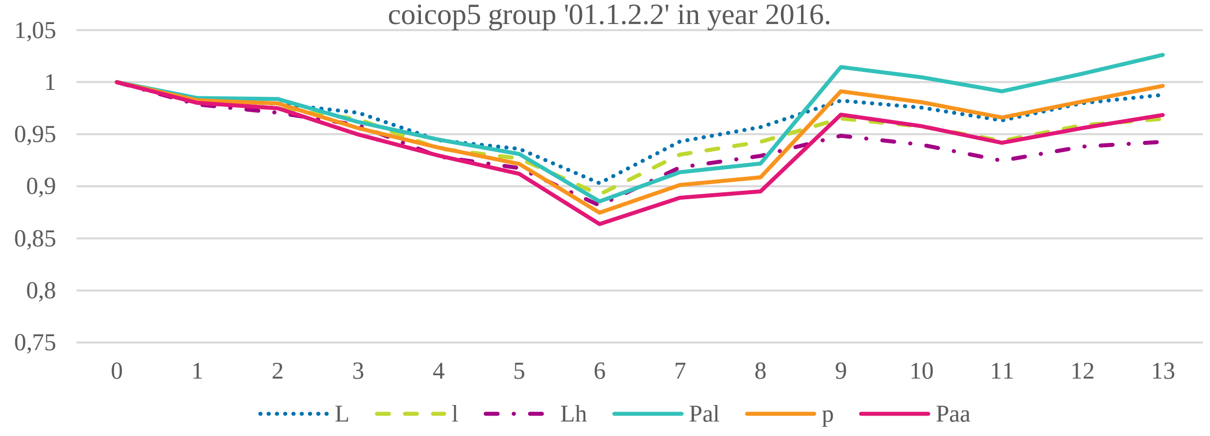


Figure 4.12: PIT for basic index number formulas for chain strategy in Finland for coicop5 group '01.1.2.3' in year 2015.

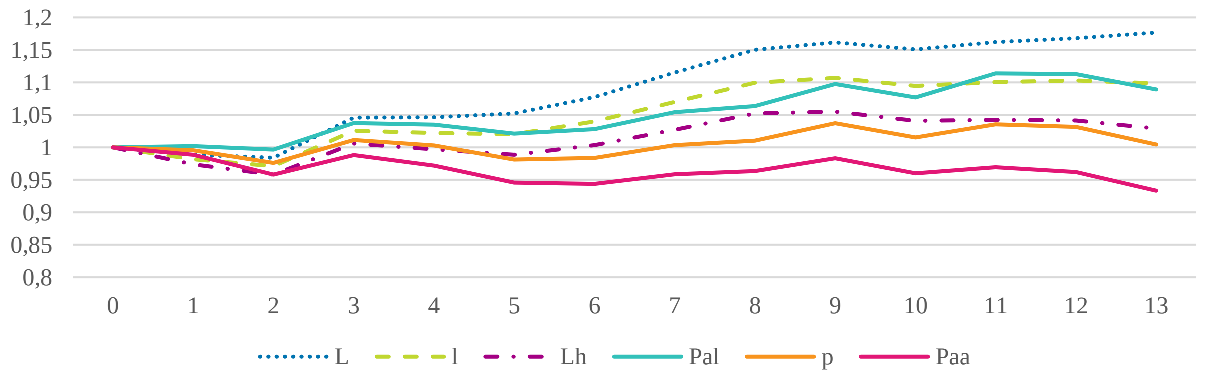


Figure 4.13: PIT for basic index number formulas for chain strategy in Finland for coicop5 group '01.1.2.3' in year 2016.

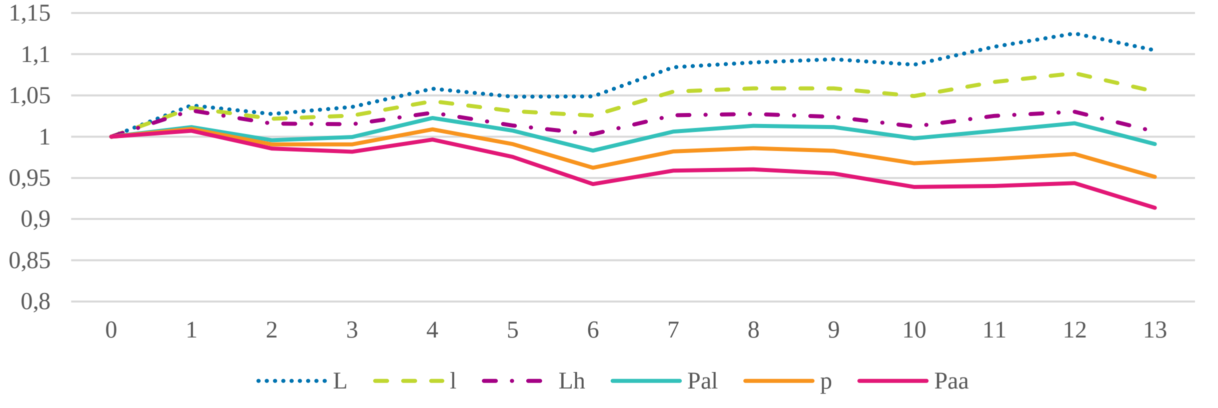


Figure 4.14: PIT for basic index number formulas for chain strategy in Finland for coicop5 group '01.1.3.1' in year 2015.

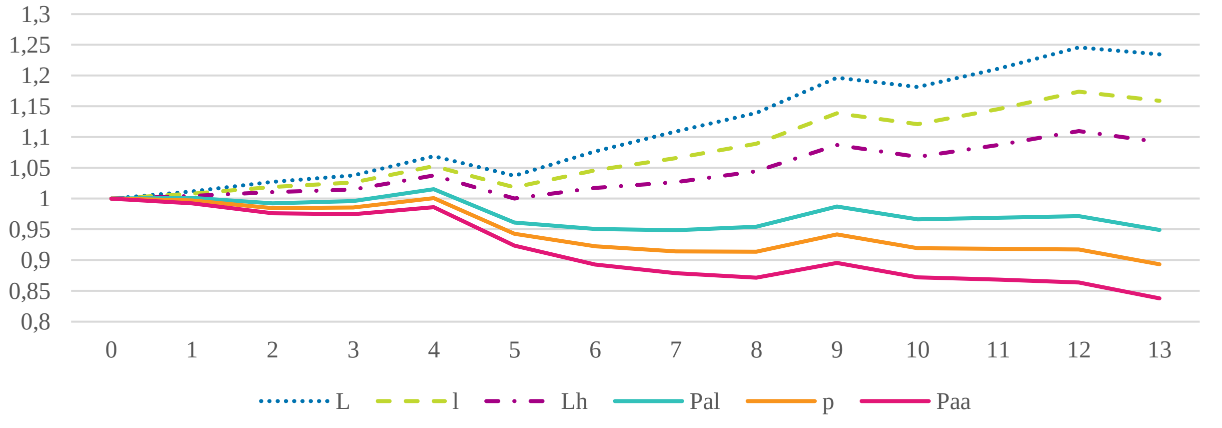
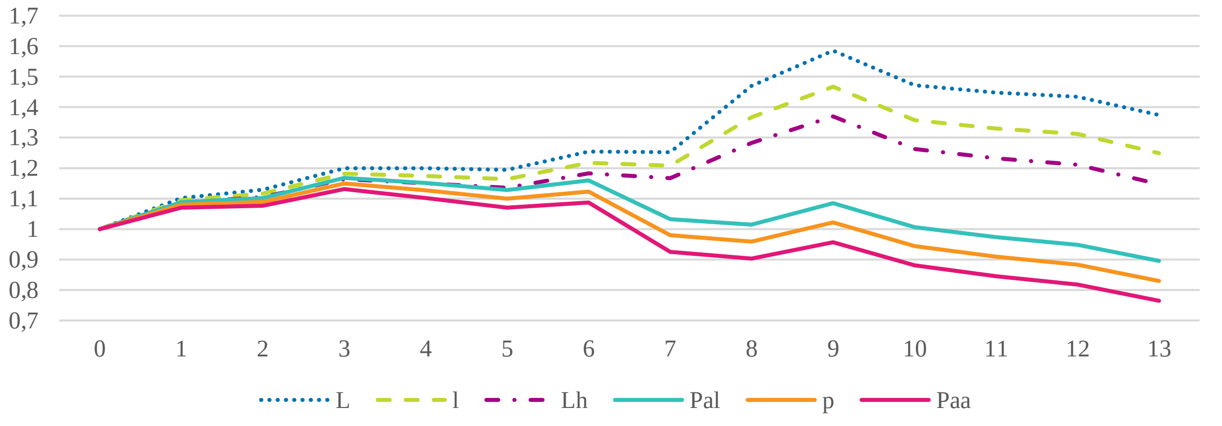


Figure 4.19: PIT for basic index number formulas for chain strategy in Finland for coicop5 group '01.1.3.1' in year 2016.



These figures show clearly how unreliable the basic indices are, when used as index formulas in chain-type strategies.

4.2.3 Path Independence Tests for Excellent Formulas in years 2015 and 2016

Finally, we show results for the path independence tests applied to the chain strategy and for the excellent index number formulas for some commodity groups. In these years 41 and 65 commodity groups from 96 have severe chain error (i.e. index series deviate more than 0.5 % form unity), commodities with harmless chain errors are presented in Appendix 1. Note that point 6 is December, from which we return to the base period 0 using odd months.

Figure 4.15: PIT for excellent index number formulas for chain strategy in Finland for coicop5 group '01.1.2.2' in year 2015.

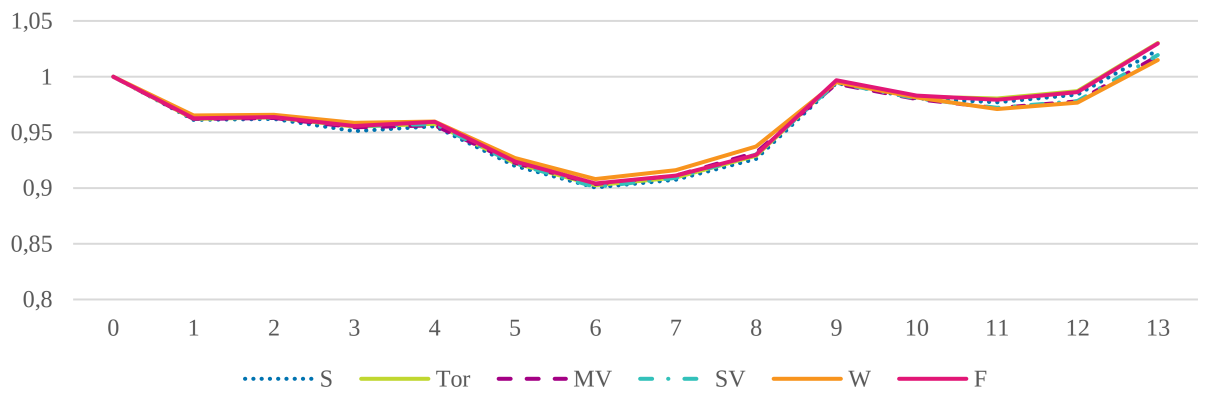


Figure 4.16: PIT for excellent index number formulas for chain strategy in Finland for coicop5 group '01.1.2.2' in year 2016.

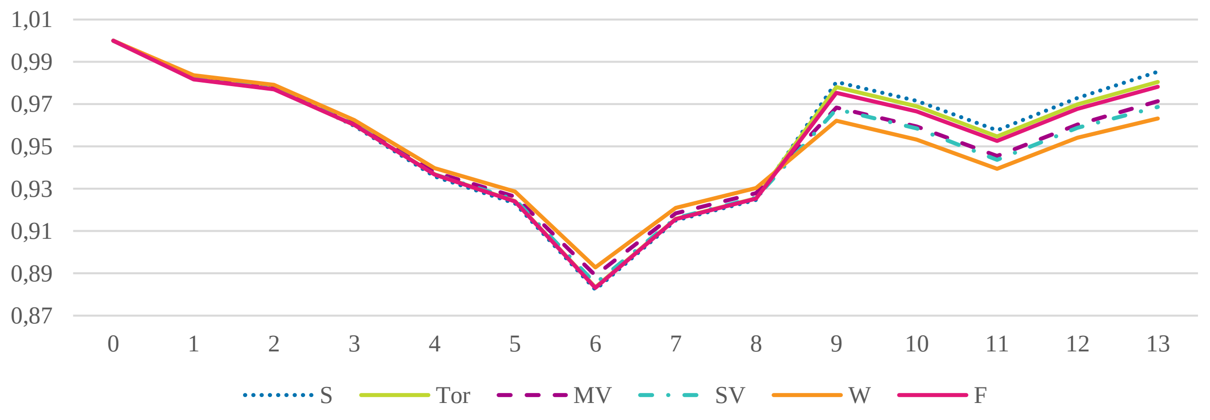


Figure 4.17: PIT for excellent index number formulas for chain strategy in Finland for coicop5 group '01.1.2.3' in year 2015.

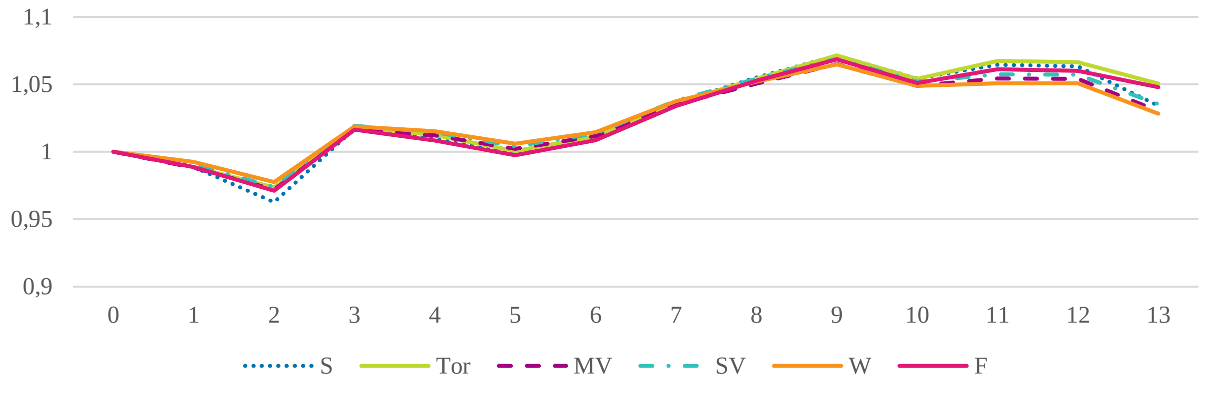


Figure 4.18: PIT for excellent index number formulas for chain strategy in Finland for coicop5 group '01.1.2.2' in year 2015.

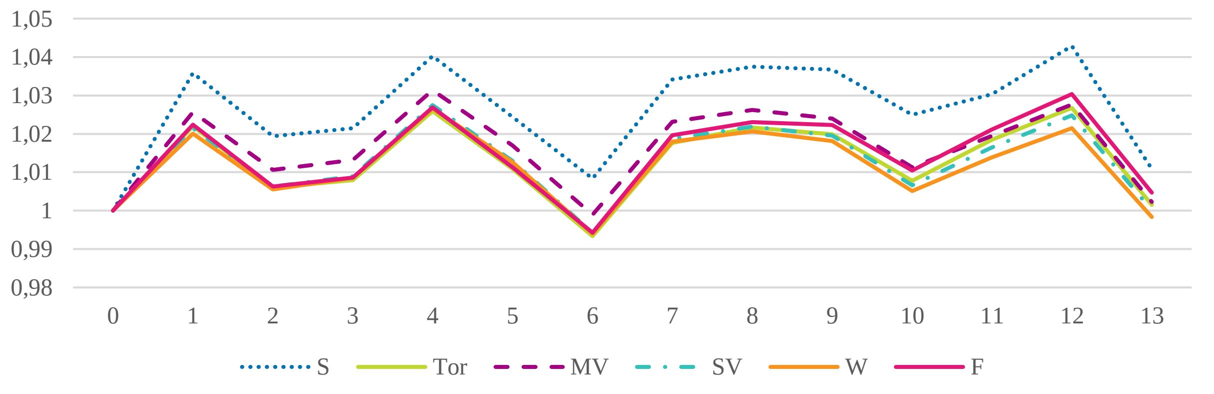
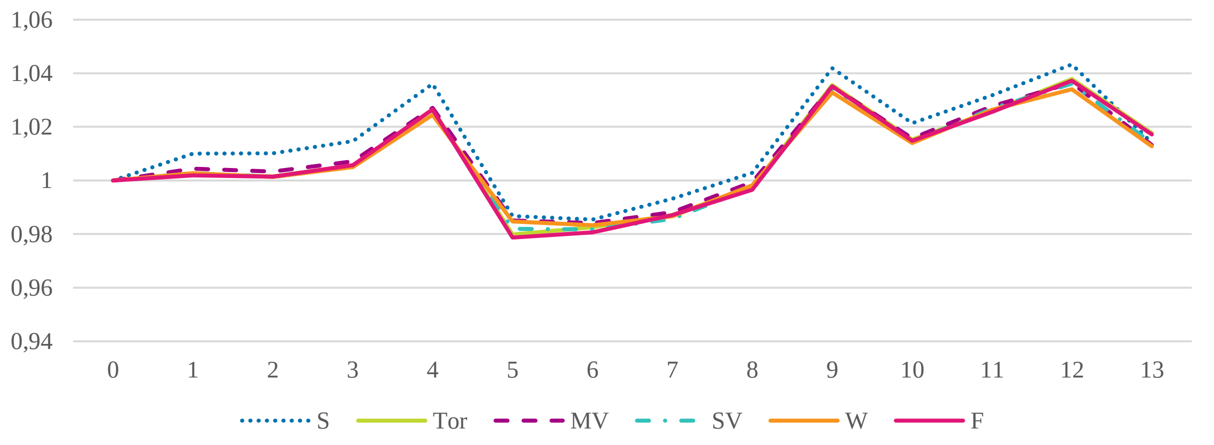
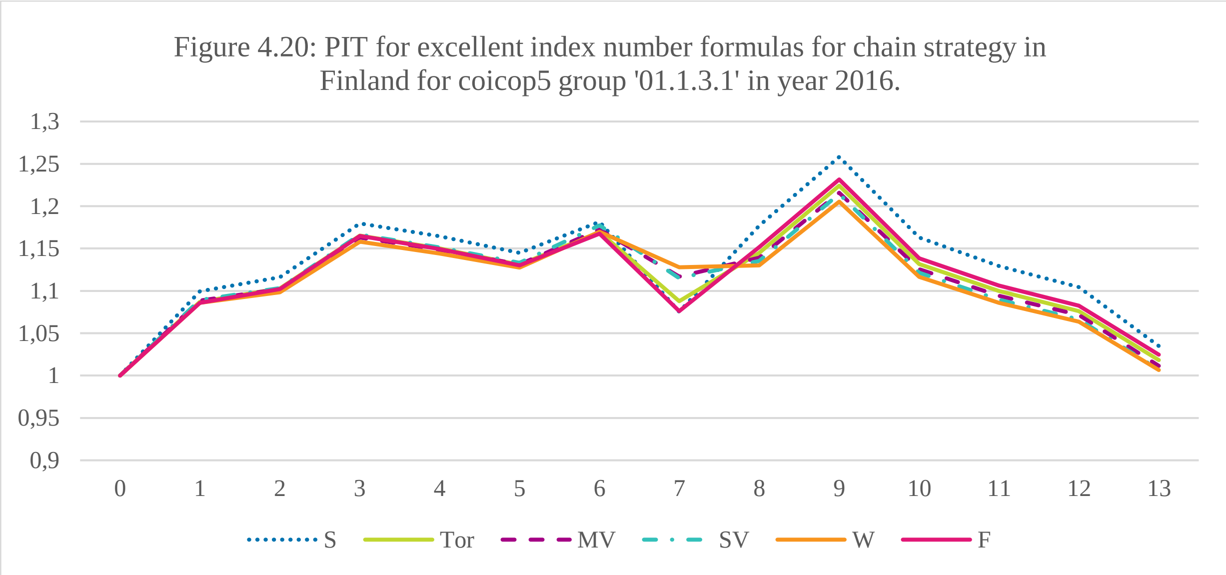


Figure 4.19: PIT for excellent index number formulas for chain strategy in Finland for coicop5 group '01.1.3.1' in year 2015.





The figures show, that the chain error is data contingent in nature. Even though the excellent formulas behave quite similarly, the cumulative chain errors are severe for the shown commodity groups.

5 Conclusions

A particular value of a price index number depends on data in question, on the strategy and on the index number formula used in index calculations. In this study we have analyzed scanner type complete micro data that contains commodity specific information of prices and quantities at all time periods. The choice of the strategy of calculation focus on different strategies and index number formulas.

We use base and chain strategies and two sets of index number formulas. The first set is based on index number formulas using old (i.e. *L*, *l*, *Lh*) or new (i.e. *Pl*, *p*, *Pa*) weights. These all are contingently biases formulas in all price-links. The second set includes six excellent formulas (i.e. *S*, *T*, *MV*, *SV*, *W*, *F*). As already was known the base strategy is free of the chain error for any index number formula. However, according to Vartia & Suoperä (2018), basic index number formulas using old or new value shares are always contingently biased even for base strategy and should never be used, if excellent formulas are available. Instead, excellent formulas and the base strategy together gives chain error free results. These choices make a good choice in the compilation of price indices.

We show in empirical tests that the chained type strategies almost always contain the chain error that is contingent on data in question; somewhere the bias is harmless and somewhere severe. There is surely one exception for that – if all direct price-links are in harmony with the same ideal consumer demand theory, then chain error remains small. We may say that with such ideal price-links, all strategies and all excellent formulas give approximately the same results, but not otherwise. For instance, strong seasonal variation (Christmas, Easter) is not compatible with the assumption of the invariant demand theory and thus chain error arises for seasonal commodities. Similarly, bouncing commodities is in variance with consumer theory and chain error probably arises.

According to empirical finding's we suggest

1. Never use a basic index or any other contingently biased index number formulas, if excellent index numbers are available. Their price-links are contingently biased, see Vartia & Suoperä, 2018. This holds both for base and chain indices (or their mixtures).
2. Choose the base strategy for the monthly index.
3. Use price-links from previous year (normalized to average month) to current year months, i.e. base strategy $(year - 1) \rightarrow year.month$.
4. The chain index on annual level, i.e. price links $(year - 1) \rightarrow year$ based on annual total values and total quantities and unit values as prices (on the homogeneous commodity level), may function much better than the chain strategies applied for months. This point needs further research.
5. Use some excellent formula, such as *MV* which is consistent in aggregation and merges null-values (new or vanishing commodities).

The base strategy in point 3 is free of chain or circular error, which would invalidate e.g. chain strategy between months. Selection of the previous year $(year - 1) \rightarrow year.month$ as a base period includes also seasonal commodities all the time and so they are included in the price-links when needed. Using excellent instead of contingently biased formulas, avoids unnecessary contingent biases.

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Appendix 1:

Table 4.1: Harmless chain error for basic index number formulas for coicop5 commodity groups in Finland and in five regions in years 2015 and 2016 (96 test groups).		
Regions	Harmless chain error in the year 2015	Harmless chain error in the year 2016
In Finland	01.1.4.3, 01.1.4.4, 01.1.4.6, 01.1.9.9	01.1.4.2, 01.1.4.4, 01.1.4.6
In region=1	01.1.4.3, 01.1.4.4, 01.1.4.6, 01.1.9.9	01.1.4.1, 01.1.4.2, 01.1.4.6, 01.1.9.9
In region=2	01.1.4.3, 01.1.4.4, 01.1.4.6, 01.1.9.9	01.1.4.1, 01.1.4.4, 01.1.4.6
In region=3	01.1.4.3, 01.1.4.4, 01.1.4.6	01.1.4.1, 01.1.4.6
In region=4	01.1.4.3, 01.1.4.4, 01.1.4.6, 01.1.9.9	01.1.4.4, 01.1.4.5, 01.1.4.6
In region=5	01.1.4.2, 01.1.4.3, 01.1.4.4, 01.1.4.6, 01.1.9.9	01.1.4.1, 01.1.4.4, 01.1.4.6, 01.1.9.9

Table 4.2: Harmless chain error for excellent index number formulas for coicop5 commodity groups in Finland and in five regions in years 2015 and 2016 (96 test groups, the MPIT's are used for the chain strategy, see p. 9 eq. (6)).		
Regions	Harmless chain error in the year 2015	Harmless chain error in the year 2016
In Finland	01.1.4.3 - 01.1.4.6, 01.1.9.9	01.1.2.6, 01.1.4.1, 01.1.4.2, 01.1.4.4 - 01.1.4.6, 01.1.9.9
In region=1	01.1.4.3 - 01.1.4.6, 01.1.9.9	01.1.2.6, 01.1.4.1, 01.1.4.2, 01.1.4.4 - 01.1.4.6, 01.1.9.9
In region=2	01.1.4.3 - 01.1.4.6, 01.1.9.9	01.1.4.1, 01.1.4.2, 01.1.4.4 - 01.1.4.6, 01.1.9.9
In region=3	01.1.4.3 - 01.1.4.6, 01.1.9.9	01.1.2.6, 01.1.4.1, 01.1.4.4 - 01.1.4.6
In region=4	01.1.4.3 - 01.1.4.6, 01.1.9.9	01.1.2.6, 01.1.4.2, 01.1.4.4 - 01.1.4.6
In region=5	01.1.4.1 - 01.1.4.7, 01.1.9.9	01.1.4.1, 01.1.4.2, 01.1.4.4 - 01.1.4.7, 01.1.9.9

Table 4.3: Harmless chain error for basic index number formulas for coicop5 commodity groups in Finland and in five regions in years 2015 and 2016 (96 test groups, the PIT's, see p. 12 eq. (8)).		
Regions	Harmless chain error in the year 2015	Harmless chain error in the year 2016
In Finland	01.1.4.1 - 01.1.4.4, 01.1.9.9	01.1.4.1, 01.1.4.2, 01.1.4.6, 01.1.9.9
In region=1	01.1.4.1 - 01.1.4.4, 01.1.4.6, 01.1.9.9	01.1.4.2, 01.1.4.4, 01.1.4.6, 01.1.9.9
In region=2	01.1.4.3, 01.1.4.4, 01.1.9.9	01.1.4.1, 01.1.4.2, 01.1.4.4, 01.1.4.6, 01.1.9.9
In region=3	01.1.4.1 - 01.1.4.4, 01.1.4.6, 01.1.9.9	01.1.4.1, 01.1.4.2 - 01.1.4.6, 01.1.9.9
In region=4	01.1.4.3, 01.1.4.4, 01.1.9.9	01.1.4.1, 01.1.4.2, 01.1.4.4, 01.1.4.6, 01.1.9.9
In region=5	01.1.4.1 - 01.1.4.4, 01.1.9.9	01.1.4.1, 01.1.4.2, 01.1.4.4 - 01.1.4.6, 01.1.9.9

Table 4.4: Harmless chain error for excellent index number formulas for coicop5 commodity groups in Finland and in five regions in years 2015 and 2016 (96 test groups, the PIT's, see p. 13 eq. (11)).

Regions	Harmless chain error in the year 2015	Harmless chain error in the year 2016
In Finland	01.1.2.1, 01.1.2.6, 01.1.4.1 - 01.1.4.7, 01.1.9.9	01.1.2.3, 01.1.2.4, 01.1.2.6, 01.1.4.1, 01.1.4.2, 01.1.4.4 - 01.1.4.7, 01.1.9.9
In region=1	01.1.2.1, 01.1.2.6, 01.1.3.1, 01.1.4.1 - 01.1.4.6, 01.1.9.9	01.1.2.3, 01.1.2.4, 01.1.2.6, 01.1.3.1, 01.1.4.1, 01.1.4.2, 01.1.4.4 - 01.1.4.7, 01.1.9.9
In region=2	01.1.2.1, 01.1.2.6, 01.1.4.2 - 01.1.4.7, 01.1.9.9	01.1.2.4, 01.1.2.6, 01.1.4.1, 01.1.4.2, 01.1.4.4 - 01.1.4.7, 01.1.9.9
In region=3	01.1.2.1, 01.1.2.6, 01.1.4.3 - 01.1.4.7, 01.1.9.9	01.1.2.6, 01.1.4.1, 01.1.4.2, 01.1.4.4 - 01.1.4.7, 01.1.9.9
In region=4	01.1.2.1, 01.1.2.6, 01.1.3.1, 01.1.4.2 - 01.1.4.6, 01.1.9.9	01.1.2.6, 01.1.4.1 - 01.1.4.6, 01.1.9.9
In region=5	01.1.2.1, 01.1.2.6, 01.1.4.1 - 01.1.4.6, 01.1.9.9	01.1.2.4, 01.1.2.6, 01.1.2.8, 01.1.3.1, 01.1.4.1, 01.1.4.2, 01.1.4.4 - 01.1.4.6, 01.1.9.9